UDC 539.3

ON ASYMPTOTICALLY EXACT EQUATIONS OF THIN PLATES OF COMPLEX STRUCTURE

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B. A. SHOIKHET
(Leningrad)
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We study the asymptotic behavior of the solution of the three-dimensional problem for a nonhomogeneous anisotropic plate of piecewise-continuous thickness when the characteristic relative thickness tends to zero. It is proved that this solution (properly normed) tends (in integral norms) to the solution of some two-dimensional equations, which in the case of an isotropic plate coincide with the classical ones. If the material of the plate has at every point an elastic plane of symmetry, parallel to the median plane, and if the plate has a symmetric structure, then we arrive to the well-known equations of anisotropic plates [1]. If however, there is no such plane, then, apparently, the obtained equations have not been given in the literature. The results of the paper give a partial answer to the question [2]: "in what sense is it possible to perform a limiting process from the three-dimensional problem of the theory of elasticity to the two-dimensional one for a plate whose cross section has angular points ?".

The problem of the limiting accuracy of the classical theory of thin plates of constant thickness has been thoroughly studied. Asymptotic expansions of the three-dimensional state of stress, whose first term is the solution of the classical theory were obtained in [3 – 6]. An estimate of the energy norm of the difference between the solutions of the three-dimensional problem and that of the classical theory was obtained in [7 – 9]. In [10 – 12] the state of stress of some anisotropic plates, including multi-layered ones, has been investigated by the method of [4].

1. We formulate the three-dimensional problem D_h. Assume that the median plane of the plate occupies the domain Ω of the variables $x = (x_1, x_2)$, Γ is the piecewise smooth boundary of Ω, and the plate occupies the domain

$$V_h = \{(x, x_3) \mid x \in \Omega, \ -ht_2(x) < x_3 < ht_1(x)\}$$
$$t_i(x) \geqslant m > 0, \qquad m = \text{const}, \qquad i = 1, 2$$

Here $h > 0$ is the ratio between the characteristic thickness and the characteristic dimension of the median plane, $t_1(x)$, $t_2(x)$ are piecewise smooth functions. By definition, $t(x)$ is piecewise smooth if the closure Ω^c of the domain Ω can be represented in the form

$$\Omega^c = \bigcup_{i=1}^{k} \Omega_i^c, \qquad \Omega_i \cap \Omega_j = \Lambda, \qquad i \neq j$$

Each domain Ω_i has a piecewise smooth boundary, $t(x)$ is infinitely differentiable in Ω_i, $i = 1, \ldots, k$, at the "joint" domains Ω_i the function $t(x)$ may have discontinuities.

The lateral surface is decomposed into two parts S_1 and S_2 $(\Gamma = \Gamma_1 \cup \Gamma_2)$, where

$$S_1 = \{(x, x_3) \mid x \in \Gamma_1, \ -ht_2(x) < x_3 < ht_1(x)\}$$
$$S_2 = \{(x, x_3) \mid x \in \Gamma_2, \ -ht_2(x) < x_3 < ht_1(x)\}$$

At S_1 the plate is rigidly fixed and at S_2 the distribution of stresses is given. We introduce the class of admissible displacements and strains

$$U = \{\mathbf{u} \mid \mathbf{u} = (u_1, u_2, u_3), \ u_i \in W_2^1(V_h), \ u_i = 0 \text{ на } S_1, \ i = 1, 2, 3\}$$
$$\varepsilon_{ii} = u_{i,i}, \quad i = 1, 2, 3, \quad \varepsilon_{ij} = u_{i,j} + u_{j,i}, \quad i \neq j, \quad i, j = 1, 2, 3$$

Here and in the following $f_{,i} \equiv \partial f / \partial x_i$, $f_{,z} \equiv \partial f / \partial z$.

We perform the change of variable $x_3 = hz$, then the domain V_h is transformed into the domain $V_1 = \{(x, z) \mid x \in \Omega, \ -t_2(x) < z < t_1(x)\}$. Let $A(x, z)$ be a 6×6 symmetric matrix, whose coefficients are measurable functions, uniformly bounded in the domain V_1, while A is uniformly positive definite in the domain V_1. We write Hooke's law in the form

$$\sigma = \varepsilon A(x, h^{-1}x_3) \tag{1.1}$$
$$\sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23}, \sigma_{33}), \qquad \varepsilon = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23}, \varepsilon_{33})$$

We emphasize the fact that the matrix A does not depend on h, i.e. the character of the anisotropy distribution is fixed along the thickness of the plate. We introduce the functional of the total energy

$$\Phi_h(\mathbf{u}) = E_h(\mathbf{u}) - L_h(\mathbf{u}) \tag{1.2}$$
$$E_h(\mathbf{u}) = \frac{1}{2} \int_{V_h} \sigma \varepsilon^* \, dx \, dx_3 = \frac{1}{2} \int_{V_h} \varepsilon A \varepsilon^* \, dx \, dx_3$$
$$L_h(\mathbf{u}) = \int_{V_h} F_i^h u_i \, dx \, dx_3 + \int_{\Omega} (p_i^h u_i^+ + q_i^h u_i^-) \, dx + \int_{S_2} f_i^h u_i \, d\Gamma \, dx_3 \tag{1.3}$$
$$u_i^+(x) \equiv u_i(x, ht_1(x)), \qquad u_i^-(x) \equiv u_i(x, -ht_2(x))$$

The asterisk denotes the transpose and we assume summation with respect to repeated indices.

Let
$$F_i^h \in L_2(V_h), \qquad p_i^h, q_i^h \in L_2(\Omega), \qquad f_i^h \in L_2(S_2) \tag{1.4}$$

The problem D_h consists in finding the minimum of the functional Φ_h in the class U.

Theorem 1 [13–17]. The problem D_h has a unique solution.

2. We compare the problem D_h with the following two-dimensional problem K_h. We introduce

$$m_i^h(x) = \int F_i^h(x, x_3) x_3 dx_3 + ht_1(x) p_i^h(x) - ht_2(x) q_i^h(x) \tag{2.1}$$
$$x \in \Omega, \ i = 1, 2$$
$$g_i^h(x) = h \left[\int F_i^h(x, x_3) \, dx_3 + p_i^h(x) + q_i^h(x) \right], \quad x \in \Omega, \ i = 1, 2$$
$$g_3^h(x) = \int F_3^h(x, x_3) \, dx_3 + p_3^h(x) + q_3^h(x), \quad x \in \Omega$$

$$M_i^h(x)=\int f_i^h(x, x_3)\, x_3\, dx_3, \quad x\in\Gamma_2,\ i=1, 2$$

$$T_i^h(x)=h\int f_i^h(x, x_3)\, dx_3, \quad x\in\Gamma_2,\ i=1, 2$$

$$T_3^h(x)=\int f_3^h(x, x_3)\, dx_3, \quad x\in\Gamma_2$$

The integrals in (2.1) are taken between the limits $[-ht_2(x), ht_1(x)]$. The functions m_1^h, m_2^h are the moments distributed on Ω, M_1^h, M_2^h are the principal moments, g_3^h is the normal load, T_3^h is the transverse force, $h^{-1}g_1^h$, $h^{-1}g_2^h$ are the tangential loads distributed over Ω, $h^{-1}T_1^h$, $h^{-1}T_2^h$ are the principal stretching forces.

We set $\quad \sigma=(\sigma_1, \sigma_2), \quad \sigma_1=(\sigma_{11}, \sigma_{22}, \sigma_{12}), \quad \sigma_2=(\sigma_{13}, \sigma_{23}, \sigma_{33})$

$$\varepsilon=(\varepsilon_1, \varepsilon_2), \quad \varepsilon_1=(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}), \quad \varepsilon_2=(\varepsilon_{13}, \varepsilon_{23}, \varepsilon_{33})$$

We represent (1.1) in the following form (A_{ij} are 3×3 matrices):

$$\sigma_1=\varepsilon_1 A_{11}+\varepsilon_2 A_{21}, \qquad A=\begin{Vmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{Vmatrix} \qquad (2.2)$$
$$\sigma_2=\varepsilon_1 A_{12}+\varepsilon_2 A_{22},$$

Making use of (2.2), we express σ_1 in terms of ε_1 and σ_2

$$\sigma_1=\varepsilon_1 B+\sigma_2 A_{22}^{-1}A_{21}, \qquad B=A_{11}-A_{12}A_{22}^{-1}A_{21} \qquad (2.3)$$

Lemma 1. The matrix B is symmetric, with coefficients bounded in V_1, uniformly positive definite in V_1, i.e.

$$\varepsilon_1 B\varepsilon_1^*\geqslant \nu\varepsilon_1\cdot\varepsilon_1^*, \qquad \forall\varepsilon_1,\ (x, z)\in V_1, \qquad \nu=\text{const} \qquad (2.4)$$

The proof follows from the properties of the matrix A and the identity

$$\varepsilon_1 B\varepsilon_1^*=(\varepsilon_1, -\varepsilon_1 A_{22}^{-1}A_{21})\, A\, (\varepsilon_1, -\varepsilon_1 A_{22}^{-1}A_{21})^*$$

We assume formally the Kirchhoff hypotheses; (a) the stresses σ_2 are small in comparison with σ_1, (b) the displacements are distributed according to the law

$$u_i(x, x_3)=v_i(x)-x_3 w_{,i}(x),\ i=1, 2,\ u_3(x, x_3)=w(x) \qquad (2.5)$$

where v_1, v_2, w are functions which depend only on x. Then

$$\varepsilon_1=\mu(w)x_3+\eta(v_1, v_2) \qquad (2.6)$$

$$\mu(w)=-(w_{,1,1}, \omega_{,2,2}, \omega_{,1,2}), \qquad \eta(v_1, v_2)=(v_{1,1}, v_{2,2}, v_{2,1}+v_{1,2}).$$

We transform (1.2), discarding the terms which contain σ_2, and discarding in (2.3) the terms containing σ_2, we substitute the obtained relation $\sigma_1=\varepsilon_1 B$ into E_h. Making use of (2.6), we obtain the functional

$$e_h(v_1, v_2, w)=\frac{1}{2}\int_{V_h}(\mu x_3+\eta)\, B\, (\mu x_3+\eta)^*\, dx\, dx_3=\frac{1}{2}\int_\Omega (h^3\mu P\mu^*+ \qquad (2.7)$$
$$h\eta P\eta^*+h^2\mu Q\eta^*+h^2\eta Q\mu^*)\, dx$$

$$P(x)=\frac{1}{2}\int_{-t_2(x)}^{t_1(x)}[B(x, z)+B(x, -z)]\, z^2\, dz,$$

$$Q(x)=\frac{1}{2}\int_{-t_2(x)}^{t_1(x)}[B(x, z)-B(x, -z)]\, z dz$$

We substitute (2.5) into (1.3). Integrating with respect to x_3 we obtain the functional

$$l_h(v_1, v_2, w) = \int_\Omega (h^{-1}g_i^h v_i + g_3^h w + m_i^h w_{,i})\,dx + \int_{\Gamma_2} (h^{-1}T_i^h v_i + T_3^h w - M_i^h w_{,i})\,d\Gamma \qquad (2.8)$$

We introduce the energy functional of the thin plate in the class of the three functions

$$\psi_h(v_1, v_2, w) = e_h(v_1, v_2, w) - l_h(v_1, v_2, w)$$

$$G = \{(v_1, v_2, w) \mid v_1, v_2 \in W_2^1(\Omega), w \in W_2^2(\Omega), v_1 = v_2 = w_{,1} = w_{,2} = w = 0 \text{ on } \Gamma_1\}$$

The problem K_h consists in minimizing the functional ψ_h in the class G.

Note 1. If the plate has a symmetric structure, i.e. $A(x, z) = A(x, -z)$, then also $B(x, z) = B(x, -z)$ and the matrix $Q(x)$ is equal to zero. Then the problem K_h splits into a bending and an extension-compression problem.

We introduce the space θ, consisting of all possible collections of functions $\vartheta = (m_1, m_2, g_1, g_2, g_3, M_1, M_2, T_1, T_2, T_3)$, such that $m_i, g_i \in L_2(\Omega)$, M_i, $T_i \in L_2(\Gamma_2)$; by $\|\vartheta\|$ we denote the sum of the norms in L_2 of all the components of ϑ. For the sake of brevity we denote the system of loads of the three-dimensional problem by N^h, then the formulas (2.1) can be considered as a transformation which associates to every system of loads N^h a system of forces and moments $\vartheta^h \in \theta$

$$\vartheta^h = (m_1^h, m_2^h, g_1^h, g_2^h, g_3^h, M_1^h, M_2^h, T_1^h, T_2^h, T_3^h)$$

Obviously, the problem K_h makes sense for any $\vartheta \in \theta$, and not only for $\vartheta = \vartheta^h$. We introduce the notation: if V is a domain, then the norms in the spaces $L_2(V)$, $W_2^1(V)$, $W_2^2(V)$ will be denoted by $\|\cdot\|_V$, $\|\cdot\|_{1,V}$, $\|\cdot\|_{2,V}$, respectively.

Lemma 2. The problem $K_h(\vartheta)$ has a unique solution (v_1^h, v_2^h, w^h), and

$$v_1^h = h^{-2}v_1^1, \qquad v_2^h = h^{-2}v_2^1, \qquad w^h = h^{-3}w^1 \qquad (2.9)$$

where (v_1^1, v_2^1, w^1) is the solution of the problem $K_1(\vartheta)$.

Proof. Since ψ_h is a convex functional, the existence and the uniqueness of the solution follows from the inequality [18]

$$(\|v_1\|_{1,\Omega}^2 + \|v_2\|_{1,\Omega}^2 + \|w\|_{2,\Omega}^2) \leqslant c e_h(v_1, v_2, w) \qquad (2.10)$$

which holds for any triplet $(v_1, v_2, w) \in G$; here and in the following we will denote by the letter c different constants. Let us prove (2.10). From (2.4)

$$e_h \geqslant \frac{\nu}{2} \int_{V_h} [\mu(w)x_3 + \eta(v_1, v_2)]\,[\mu(w)x_3 + \eta(v_1, v_2)]^* \,dx dx_3 \geqslant$$
$$\frac{h^3 m^3 \nu}{3} \int_\Omega [(w_{,1,1})^2 + 4(w_{,1,2})^2 + (w_{,2,2})^2]\,dx +$$
$$\nu h m \int_\Omega [(v_{1,1})^2 + (v_{2,2})^2 + (v_{2,1} + v_{1,2})^2]\,dx$$

Since w, $w_{,1}$, $w_{,2}$ vanish at Γ_1, the first of the integrals of the right-hand side is a majorant of the norm w in $W_2^2(\Omega)$. Korn's inequality is well-known [13 − 17]; let $V \in R^n$ be a domain with a piecewise smooth boundary and assume that on some $(n-1)$-

dimensional piece ω of the boundary of the domain V the functions $v_1, v_2, \ldots, v_n$, belonging to $W_2^1(V)$, are equal to zero, then

$$\sum_{i=1}^{n} \| v_i \|^2_{1,V} \leqslant c \int_V \Big[\sum_{i,j=1}^{n} (v_{i,j} + v_{j,i})^2 \Big] dV \tag{2.11}$$

with a constant c depending on V and ω. From (2.11) we obtain (2.10).

We can verify that $h^3\psi_h(h^{-2}v_1, h^{-2}v_2, h^{-3}w) = \psi_1(v_1, v_2, w)$, from where we obtain (2.9).

We require that N^h satisfy the condition

$$\begin{aligned} &\| p_i^h, q_i^h \|_\Omega \leqslant h^{-1}c_0, \quad i = 1, 2, \quad \| p_3^h, q_3^h \|_\Omega \leqslant c_0, \\ &\| f_i^h \|_{S_2} \leqslant h^{-3/2}c_0, \quad i = 1, 2 \\ &\| f_3^h \|_{S_2} \leqslant h^{-1/2}c_0, \quad \| F_i^h \|_{V_h} \leqslant h^{-3/2}c_0, \quad i = 1, 2, \quad \| F_3^h \|_{V_h} \leqslant h^{-1/2}c_0 \end{aligned} \tag{2.12}$$

where the constant c_0 does not depend on h. Let us consider the sense of the conditions (2.12). Each of the loads creates a state of stress which tends to infinity with a well-defined rate. For the fundamental load we have chosen the load normal to the upper face, the corresponding state of stress behaving like: $u_1, u_2, \sigma_{11}, \sigma_{22}, \sigma_{12} \sim h^{-2}$, $u_3 \sim h^{-3}$. Therefore, if the tangential loads $p_1^h, p_2^h, q_1^h, q_2^h$ increase whith the change of h, but with a rate not higher than h^{-1} (as prescribed by the conditions (2.12)), their contribution to the state of stress has an order of growth not higher than the contribution of the normal load. The remaining estimates were selected from similar considerations. It can be verified that if N^h satisfies the conditions (2.12), then

$$\| \vartheta^h \| \leqslant cc_0, \ \text{ i.e. } \ \| m_i^h, g_i^h \|_\Omega \leqslant cc_0, \ \| M_i^h, T_i^h \|_{\Gamma_2} < cc_0 \tag{2.13}$$

where the constant c does not depend on h.

Theorem 2. Assume that N^h satisfies condition (2.12) and that we have for $h \to 0$ the limit

$$\vartheta^h \to \vartheta^\circ, \ \vartheta^\circ = (m_1^\circ, m_2^\circ, g_1^\circ, g_2^\circ, g_3^\circ \ M_1^\circ, M_2^\circ, T_1^\circ, T_2^\circ, T_3^\circ)$$

(This means that each component of ϑ^h converges in L_2 to the corresponding component of ϑ°). We denote by (v_1^1, v_2^1, w^1) the solution of the problem $K_1(\vartheta^\circ)$, then the solution u^h, σ_{ij}^h of the problem $D_h(N^h)$ can be represented in the form

$$\begin{aligned} &u_i^h(x, x_3) = h^{-2}[-w_{,i}^1(x)h^{-1}x_3 + v_i^1(x) + R_i^h(x, h^{-1}x_3)], \quad i = 1, 2 \\ &u_3^h(x, x_3) = h^{-3}[w^1(x) + R_3^h(x, h^{-1}x_3)] \\ &\boldsymbol{\sigma}_1^h(x, x_3) = (\sigma_{11}^h, \sigma_{22}^h, \sigma_{12}^h) = \\ &\qquad h^{-2}\{B[\boldsymbol{\mu}(w^1)h^{-1}x_3 + \boldsymbol{\eta}(v_1^1, v_2^1)] + \mathbf{R}_1^h(x, h^{-1}x_3)\} \\ &\boldsymbol{\sigma}_2^h(x, x_3) = (\sigma_{13}^h, \sigma_{23}^h, \sigma_{33}^h) = h^{-2}\mathbf{R}_2^h(x, h^{-1}x_3) \\ &(\| R_i^h \|_{1,V_1} \to 0, \ i = 1, 2, 3, \quad \| \mathbf{R}_i^h \|_{V_1} \to 0, \quad i = 1, 2 \quad \text{for} \quad h \to 0) \end{aligned} \tag{2.14}$$

Note 2. The representation (2.14) can be considered as the mathematical proof of Kirchhoff's hypothesis.

Note 3. We set

$$\omega^h(c_0, N^h, \vartheta^\circ) = \sum_{i=1}^{3} \| R_i^h \|_{1,V_1} + \sum_{i=1}^{2} \| \mathbf{R}_i^h \|_{1,V_1}$$

In Theorem 2 it is asserted that if $\vartheta^h \to \vartheta^\circ$, $h \to 0$, then $\omega^h(c_0, N^h, \vartheta^\circ) \to 0$, $h \to 0$, i.e. $\omega^h(c_0, N^h, \vartheta^\circ)$ is an estimate of the nearness of the solution of the problem $D_h(N^h)$ to the solution of the problem $K_h(\vartheta^\circ)$. Naturally it is desirable not to require $\vartheta^h \to \vartheta^\circ$, $h \to 0$, but rather compare the solution of the problem $D_h(N^h)$ directly with the solution of the problem $K_h(\vartheta^h)$, i.e. to study the behavior of the quantity $\omega^h \cdot (c_0, N^h, \vartheta^h)$. It turns out that $\omega^h(c_0, N^h, \vartheta^h) \to 0$, regardless of the behavior of ϑ^h; more exactly, the following theorem holds.

Theorem 3. We fix h and from all the loads satisfying the condition (2.12), we select the load N_*^h so that the quantity $\omega^h(c_0, N^h, \vartheta^h)$ be maximized (we can prove that such N_*^h exists)

$$\omega_*^h(c_0) = \max_{N^h} \omega^h(c_0, N^h, \vartheta^h) \equiv \omega^h(c_0, N_*^h, \vartheta_*^h)$$

Then $\omega^h_*(c_0) \to 0$, $h \to 0$.

Corollary. Let N_1^h, N_2^h be statically equivalent loads, i.e. the same ϑ^h corresponds to them, let $\mathbf{u}_1^h$, $\mathbf{u}_2^h$ be the solutions of the problems $D_h(N_1^h)$, $D_h(N_2^h)$, respectively. Then, since each of these solutions differs from the solution of the problem $K_h(\vartheta^h)$ by at most $\omega_*^h(c_0)$, they differ among themselves by at most $2\omega_*^h(c_0)$. This assertion can be considered as a weak form of the Saint-Venant principle for plates (weak in the sense that one makes use of integral norms and there is no estimate of the rate of convergence).

3. We shall divide the proof of Theorem 2 into a series of lemmas. We formulate in a different way the results of Theorem 2. Let

$$\mathbf{u} \in U, \quad U_i(x, z) = h^2 u_i(x, hz), \quad i = 1, 2, \quad U_3(x, z) = h^3 u_3(x, hz)$$
$$U_i^h(x, z) = h^2 u_i^h(x, hz), \quad i = 1, 2, \quad U_3^h(x, z) = h^3 u_3^h(x, hz)$$
$$\mathbf{U} = (U_1, U_2, U_3), \qquad \mathbf{U}^h = (U_1^h, U_2^h, U_3^h)$$

Obviously, $\mathbf{U}$, $\mathbf{U}^h$ are vectors with components from $W_2^1(V_1)$ and $U_i = U_i^h = 0$, $i = 1, 2, 3$, on $S_1^1 = \{(x, z) \mid x \in \Gamma_1, \ -t_2(x) < z < t_1(x)\}$. We introduce the quantities

$$\delta_{ii} = U_{i,i}, \quad i = 1, 2, \qquad \delta_{12} = U_{2,1} + U_{1,2} \tag{3.1}$$
$$\delta_{i3} = \delta_{3i} = h^{-1}(U_{3,i} + U_{i,z}), \quad i = 1, 2, \qquad \delta_{33} = h^{-2}U_{3,z}$$
$$\boldsymbol{\delta} = (\delta_{11}, \delta_{22}, \delta_{12}, \delta_{13}, \delta_{23}, \delta_{33}), \quad \boldsymbol{\delta}_1 = (\delta_{11}, \delta_{22}, \delta_{12}), \quad \boldsymbol{\delta}_2 = (\delta_{13}, \delta_{23}, \delta_{33})$$

We can verify that

$$\delta_{ij}(x, z) = h^2 \varepsilon_{ij}(x, hz) \tag{3.2}$$

We set

$$\delta_{ij}^h \equiv \delta_{ij}(U^h) \equiv h^2 \varepsilon_{ij}^h(x, hz), \quad \alpha_{ij}^h \equiv h^2 \sigma_{ij}^h(x, z) \tag{3.3}$$
$$\boldsymbol{\alpha}^h = (\alpha_{11}^h, \alpha_{22}^h, \alpha_{12}^h, \alpha_{13}^h, \alpha_{23}^h, \alpha_{33}^h), \quad \boldsymbol{\alpha}_1^h = (\alpha_{11}^h, \alpha_{22}^h, \alpha_{12}^h),$$
$$\boldsymbol{\alpha}_2^h = (\alpha_{13}^h, \alpha_{23}^h, \alpha_{33}^h)$$

The relations (2.14) are equivalent to the representation

$$U_i^h(x, z) = [v_i^1(x) - w_{,i}^1(x) z + R_i^h(x, z)], \quad i = 1, 2 \tag{3.4}$$
$$U_3^h(x, z) = [w^1(x) + R_3^h(x, z)]$$
$$\boldsymbol{\alpha}_1^h(x, z) = B[\boldsymbol{\mu}(w^1)z + \boldsymbol{\eta}(v_1, v_2)] + \mathbf{R}_1^h(x, z)$$
$$\boldsymbol{\alpha}_2^h(x, z) = \mathbf{R}_2^h(x, z)$$
$$(\| R_i^h \|_{1, V_1} \to 0, \ i = 1, 2, 3, \ \ \|\mathbf{R}_i^h\|_{V_1} \to 0, \ i = 1, 2, \quad \text{for} \ h \to 0)$$

Lemma 3. The following estimates uniform in h :

$$\| U_i^h \|_{1, V_1} \leqslant cc_0, \qquad i = 1, 2, 3 \tag{3.5}$$

$$\| \delta_{ij}^h \|_{V_1} \leqslant cc_0, \qquad i, j = 1, 2, 3 \tag{3.6}$$

where c_0 is the constant from the conditions (2.12), are valid.

Proof. In order that $\mathbf{u}^h$ be the solution of the problem D_h, it is necessary and sufficient that for every $u \in U$ the identity

$$\int_{V_h} \sigma^h \varepsilon^* (\mathbf{u})\, dx\, dx_3 = L_h (\mathbf{u}) \tag{3.7}$$

is satisfied. Obviously

$$\alpha^h = \delta^h A\ (x, z) \tag{3.8}$$

Multiplying both sides of (3.7) by h^3 and making use of (3.1) – (3.3), we obtain the identity

$$\int_{V_1} \alpha^h \delta^* (\mathrm{U})\, dx\, dz = S (\mathrm{U}) \tag{3.9}$$

$$S (\mathrm{U}) = \int_{V_1} \Big(h^2 \sum_{i=1}^{2} F_i^h U_i + h F_3^h U_3\Big)\, dx\, dz + \int_{S_2^1} h^2 \sum_{i=1}^{2} f_i^h U_i\, d\Gamma\, dz +$$

$$\int_{S_2^1} h f_3^h U_3\, d\Gamma\, dz + \int_{\Omega} \Big[h \sum_{i=1}^{2} (p_i^h U_i^+ + q_i^h U_i^-) + p_3^h U_3^+ + q_3^h U_3^-\Big] dx$$

or, in another form, the identity

$$\int_{V_1} \delta^h A \delta^* (\mathrm{U})\, dx\, dz = S (\mathrm{U}) \tag{3.10}$$

The estimates (2.12) have been chosen in such a way that, making use of Cauchy inequality and the imbedding theorem [19], we obtain

$$| S\ (\mathrm{U}) | \leqslant cc_0 \| U \|_{1, V_1}$$

Substituting $\mathrm{U} = \mathrm{U}^h$, into (3.10), we obtain the estimate

$$\int_{V_1} \sum_{i \leqslant j} (\delta_{ij}^h)^2\, dx\, dz = \int_{V_1} \Big[\sum_{i=1}^{2} (U_{i,i}^h)^2 + (U_{2,1} + U_{1,2})^2 + \tag{3,11}$$

$$\sum_{i=1}^{2} h^{-2} (U_{3,i}^h + U_{i,z}^h)^2 + h^{-4} (U_{3,z}^h)^2\Big]\, dx\, dz \leqslant cc_0 \| \mathrm{U}^h \|_{1, V_1}$$

Let $h < 1$, we strengthen the inequality (3.11), replacing in the left-hand side h by unity, making use of Korn inequality (2.11) and Cauchy inequality with ε, we obtain (3.5) and from (3.5), (3.11) the estimate (3.6) follows.

Corollary. It follows from (3.5) that the family U^h is weakly compact in $W_2^1(V_1)$, while from (3.6), (3.8) it follows that the families δ_{ij}^h, α_{ij}^h are weakly compact in $L_2\ (V_1)$.

We denote by $\mathrm{U}^\circ = (U_1^\circ, U_2^\circ, U_3^\circ)$, δ_{ij}°, α_{ij}° some weak limit points of these families. Later we will prove the uniqueness of the limit, therefore, without loss of generality we assume that U_i^h converges weakly to U_i° in $W_2^1\ (V_1)$, δ_{ij}^h converges weakly to δ_{ij}° in $L_2\ (V_1)$, and α_{ij}^h converges weakly to α_{ij}° in $L_2\ (V_1)$.

Lemma 4. The functions U_1°, U_2°, U_3° can be represented in the form

$$U_3^\circ (x, z) \equiv U_3^\circ (x), \qquad U_3^\circ \in W_2^2 (\Omega) \tag{3.12}$$

$$U_i^\circ (x, z) = V_i^\circ (x) - U_{3,i}^\circ (x)z, \qquad V_i^\circ (x) \in W_2^1 (\Omega), \quad i = 1, 2 \tag{3.13}$$

$$V_1^\circ = V_2^\circ = U_{3,1}^\circ = U_{3,2}^\circ = U_3^\circ = 0 \text{ on } \Gamma_1, \text{ i.e. } (V_1^\circ, V_2^\circ, U_3^\circ) \in G \tag{3.14}$$

Proof. From (3.6) we have $\| U^h_{3,z} \|_{V_1} \leqslant cc_0 h^2$, therefore $U^\circ_{3,z} \equiv 0$ in V_1, i.e.(3.12) holds. From (3.6) we have $\| U^h_{3,i} + U^h_{i,z} \|_{V_1} \leqslant cc_0 h$, $i = 1, 2$, whence $U^\circ_{i,z} = -U^\circ_{3,i}$, therefore denoting by $V_i^\circ (x)$ the trace of the function U_i° on the plane $\{x \in \Omega, z = 0\}$, we obtain (3.13), where $V_i^\circ (x) \in L_2 (\Omega)$. In the left-hand side of (3.13) we have a function which belongs to $W_2^1 (V_1)$ and is equal to zero on S_1^1, therefore it is necessary that V_i°, $U^\circ_{3,i} \in W_2^1 (\Omega)$ and (3.14) holds.

Corollary.

$$\delta_1^\circ = \mu (U_3^\circ) z + \eta (V_1^\circ, V_1^\circ) \tag{3.15}$$

Lemma 5. The following equalities are valid:

$$\alpha_{13}^\circ = 0, \alpha_{23}^\circ = 0, \alpha_{33}^\circ = 0, \quad \text{i.e. } \alpha_2^\circ = 0 \tag{3.16}$$

Proof. Let us prove first that $\alpha_{13}^\circ, \alpha_{23}^\circ, \alpha_{33}^\circ$ are constants with respect to the coordinate z. Let φ be a smooth finite function in the domain V_1. We set in the identity (3.9) $\mathbf{U} = (0, 0, \varphi)$, then

$$\int_{V_1} (\alpha_{13}{}^h h^{-1} \varphi_{,1} + \alpha_{23}{}^h h^{-1} \varphi_{,2} + \alpha_{33}{}^h h^{-2} \varphi_{,z})\, dx\, dz = S (\mathbf{U}) \tag{3.17}$$

Multiplying (3.17) by h^2 and taking the limit for fixed φ and $h \to 0$, we obtain the identity

$$\int_{V_1} \alpha_{33}^\circ \varphi_{,z}\, dx\, dz = 0$$

Let us prove that if $f \in L_2 (V_1)$ and if for every smooth finite function φ in V_1 the identity

$$\int_{V_1} f \varphi_{,z}\, dx\, dz = 0 \tag{3.18}$$

is valid, then $f (x, z) \equiv f (x)$. In fact, assume that the positive number ρ is smaller than the distance from the boundary of V_1 to the carrier of the function ψ (the carrier of a function is the set of points where the function is different from zero), then changing in (3.19) φ by φ_ρ (φ_ρ is the average of the function φ in the sense of S. L. Sobolev), we obtain

$$\int_{V_1} f (\varphi_\rho)_{,z} dx\, dz = - \int_{V} (f_\rho)_{,z}\, \varphi\, dx\, dz = 0$$

Hence it follows that in each interior subdomain of the domain V_1 we have $(f_\rho)_{,z} = 0$, i.e. $f_\rho (x, z) \equiv f_\rho (x)$. But $\| f_\rho - f \|_{V_1} \to 0$ for $\rho \to 0$, from where it follows that in every interior subdomain (and, consequently, in the entire domain V_1) $f \equiv f (x)$. From what we have proved it follows that $\alpha_{33}^\circ \equiv \alpha_{33}^\circ (x)$; similarly, $\alpha_{13}^\circ \equiv \alpha_{13}^\circ (x)$, $\alpha_{23}^\circ \equiv \alpha_{23}^\circ (x)$.

Assume now that $\varphi (x)$ is a smooth finite function in the domain Ω. We set $\mathbf{U} = (0, 0, \varphi (x) z)$ in (3.9) and we obtain the identity

$$\int_{V_1} (\alpha_{13}{}^h h^{-1} z \varphi_{,1} + \alpha_{23}{}^h h^{-1} z \varphi_{,2} + \alpha_{33}{}^h h^{-2} \varphi)\, dx\, dz = S (\mathbf{U}) \tag{3.19}$$

Multiplying (3.19) by h^2 and taking the limit for $h \to 0$, we obtain

$$\int_{V_1} \alpha_{33}^\circ \varphi \, dx \, dz = 0 \tag{3.20}$$

But α°_{33} depends only on x, therefore from (3.20) it follows that $\alpha_{33}^\circ \equiv 0$. Similarly, we obtain that $\alpha_{13}^\circ = \alpha_{23}^\circ = 0$.

Lemma 6. The following equalities are valid:

$$V_1^\circ = v_1^1, \quad v_2^\circ = v_2^1, \quad U_3^\circ = w^1 \tag{3.21}$$

Proof. Let us prove that the triplet $(V_1^\circ, V_2^\circ, U_3^\circ)$ satisfies the identity (3.24), which is a necessary and sufficient condition in order that this triplet of functions be the solution of the problem $K_1(\vartheta^\circ)$, and by virtue of the uniqueness of the solution of problem $K_1(\vartheta^\circ)$ we obtain (3.21).

We set in (3.9) $\mathrm{U} = (v_1 - w_{,1}z_1, \; v_2 - w_{,2}z, \; w)$, where $(v_1, v_2, w) \in G$. Then $\delta_1(\mathrm{U}) = \mu(w)z + \eta(v_1, v_2)$, $\delta_2(\mathrm{U}) = 0$, and (3.9) takes the form

$$\int_{V_1} \alpha_1^h \delta_1^*(\mathrm{U}) \, dx \, dz = S(\mathrm{U}) \tag{3.22}$$

But from (3.8), (3.3), (2.3) it follows that

$$\alpha_1^h = \delta_1^h B + \alpha_2^h A_{22}^{-1} A_{21} \tag{3.23}$$

Substituting (3.23) into (3.22), taking the limit for $h \to 0$, making use of (3.16), (3.15) and integrating with respect to z, we obtain the identity

$$\int_\Omega (\mu_0 P \mu^* + \eta_0 P \eta^* + \mu_0 Q \eta^* + \eta_0 Q \mu^*) \, dx = l_1(v_1, v_2, w) \tag{3.24}$$

$$(\mu \equiv \mu(w), \quad \eta \equiv \eta(v_1, v_2), \quad \mu_0 \equiv \mu(U_3^\circ), \quad \eta_0 \equiv \eta(V_1^\circ, V_2^\circ))$$

which proves the lemma.

Thus, the representations (3.4) are proved with the stipulation that R_i^h, $\mathbf{R}_i^h$ converge weakly to zero in $W_2^1(V_1)$, $L_2(V_1)$, respectively.

Lemma 7. [20]. Let H be a Hilbert space. If the sequence $\{f_h\} \in H$ and f_h converges weakly to f_0 in H and, in addition, $\| f \|_H \to \| f_0 \|_H$, $h \to 0$, then f_h converges strongly to f_0 in H.

Lemma 8 (concluding the proof of Theorem 2). The following limits for $h \to 0$ are valid:

$$\| U_i^h - U_i^\circ \|_{1, V_1} \to 0, \qquad i = 1, 2, 3 \tag{3.25}$$

$$\| \alpha_2^h \|_{V_1} \to 0 \tag{3.26}$$

Proof. Substituting $\mathrm{U} = \mathrm{U}^h$ into (3.10), after transformations we obtain

$$\int_{V_1} [\delta_1^h B (\delta_1^h)^* + \alpha_2^h A_{22}^{-1} (\alpha_2^h)^*] \, dx \, dz = S(\mathrm{U}^h) \tag{3.27}$$

By virtue of the embedding theorem [19], the family U^h is strongly compact in $L_2(V_1)$, while the family of the traces of the functions U^h on the surface S_2^1 is strongly compact in $L_2(S_2^1)$. Therefore, making use of (1.4), (2.1), (2.12), we can show that for $h \to 0$

$$S(\mathrm{U}^h) \to l_1(V_1^\circ, V_2^\circ, U_3^\circ) = \int_{V_1} \delta_1^\circ B (\delta_1^\circ)^* \, dx \, dz \tag{3.28}$$

Since $\delta_1{}^h$ converges weakly in $L_2(V_1)$ to $\delta_1{}^\circ$, we have the inequality

$$\int_{V_1} \delta_1{}^\circ B(\delta_1{}^\circ)^*\, dx\, dz \leqslant \lim_{h\to 0} \int_{V_1} \delta_1{}^h B(\delta_1{}^h)^*\, dx\, dz \qquad (3.29)$$

From (3.27) – (3.29) and from the uniform positive definiteness of the matrix $A_{22}{}^{-1}$ it follows that

$$\int_{V_1} \delta_1{}^h B(\delta_1{}^h)^*\, dx\, dz \to \int_{V_1} \delta_1{}^\circ B(\delta_1{}^\circ)^*\, dx\, dz, \quad h\to 0 \qquad (3.30)$$

$$\int_{V_1} \alpha_2{}^h A_{22}^{-1}(\alpha_2{}^h)^*\, dx\, dz \to 0, \quad h\to 0 \qquad (3.31)$$

From (3.31) we obtain (3.26), while from (3.30) and from Lemma 7 we obtain

$$\|\delta_1{}^h - \delta_1{}^\circ\|_{V_1} \to 0, \qquad h\to 0 \qquad (3.32)$$

Lemma 4, (3.6), (3.32) give the following relations (in (3.33) the convergence is in $L_2(V_1)$ for $h\to 0$):

$$(U_{3,i}^h + U_{1,z}^h) \to 0 \equiv \overset{\circ}{U}_{3,i} + \overset{\circ}{U}_{i.z}, \quad i=1,2; \quad U_{3,z}^h \to 0 \equiv \overset{\circ}{U}_{3,z} \qquad (3.33)$$

$$U_{i,i}^h \to \overset{\circ}{U}_{i,i}, \quad i=1,2; \quad (U_{1,2}^h + U_{2,1}^h) \to (\overset{\circ}{U}_{1,2} + \overset{\circ}{U}_{2,1})$$

From (3.33) and Korn's inequality (2.11) we obtain (3.25).

Let us prove Theorem 3. Assume that the assertion of the theorem does not hold. By $\vartheta_*{}^h$ we denote the totality of the integral characteristics corresponding to the load $N_*{}^h$. By virtue of (2.13) there exists a sequence $h_i\to 0$, $i\to\infty$, and $\vartheta^\circ \in \Theta$, such that each of the components ϑ^{h_i} converges weakly in L_2 to the corresponding component of ϑ°. Observing the proof of Theorem 2, we can see that in order that it should hold it is sufficient to have at least the weak convergence of ϑ^h to ϑ°, therefore $\omega^{h_i}(c_0, N_*^{h_i}, \vartheta^\circ) \to 0$, $i\to\infty$. Following the proof of Lemma 8, we can prove that

$$|\,\omega^{h_i}(c_0, N_*^{h_i}, \vartheta^\circ) - \omega^{h_i}(c_0, N_*^{h_i}, \vartheta_*^{h_i})\,| \to 0, \quad i\to\infty,$$

and then we obtain that $\omega^{h_i}{}_*(c_0) \to 0$, $i\to\infty$. The contradiction obtained proves the theorem.

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CONTACT PROBLEM OF ROLLING OF A VISCOELASTIC CYLINDER ON A BASE OF THE SAME MATERIAL

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I. G. GORIACHEVA
(Moscow)
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The problem of rolling of a viscoelastic cylinder on a base of the same material is solved under the assumption that the whole contact area consists of two sections: a section with adhesion and a section with slipping of the contacting surfaces. Equations are found to determine the length of the contact area and the adhesion section, as are expressions for the stresses on the contact area.